Reduced Form of the Green's Functions for Disks and Annular Rings

Fayez A. Alhargan and Sunil R. Judah

Abstract—**Available Green's functions for circular and annular ring microstrip circuits involve doubly infinite series. These series are computationally expensive in terms of the time necessary for summing the series and the memory required to hold the eigenvalues. In this paper the Green's function is simplified to a single series using a new single-summation method. The resulting single series eliminates the need for the eigenvalues and increases the speed of computation.**

I. Introduction

In the analysis of microstrip antennas of patch circuits the Green's function is obtained using the modal expansion. This method gives rise to a doubly infinite series [1]. The mode matching method has also been used to obtain the impedance directly in a single series format; this method is applicable to ports around the periphery only. However, Carslaw [2] has used the result of Kneser [5] to show that the Green's function for cylindrical coordinates can be analytically summed over the modes, reducing the Green's function by one series. In this paper the Green's function is simplified by a method different from that of [5]. Although there are a number of methods for obtaining this result, the method used here is direct in its approach. The Green's functions for the circular disk and the annular ring are obtained in single series format and the two methods are compared for both accuracy and efficiency.

II. Green's Function for a Circular Disk

The Green's function for a circle of radius a is given by [1]

$$G(r,\phi/r_0,\phi_0) = -\frac{j\omega\mu d}{\pi k^2a^2} + \frac{j\omega\mu d}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \cdot \frac{\sigma_n J_n(r_0k_{nm})J_n(rk_{nm})\cos n(\phi-\phi_0)}{\left(a^2-\frac{n^2}{k_{nm}^2}\right)(k_{nm}^2-k^2)J_n^2(ak_{nm})}. \quad (1)$$

This can be simplified to

$$G(r,\phi/r_0,\phi_0) = \frac{j\omega\mu d}{4}\sum_{n=0}^{\infty} \cdot \frac{\sigma_n J_n(rk)f(kr_0,ka)\cos n(\phi-\phi_0)}{J_n'(ka)}. \quad (2)$$

$$0 \leqslant r \leqslant r_0 \leqslant a, \qquad r_0 \neq 0$$

where

$$f(rk,ak) = Y_n'(ak)J_n(rk) - J_n'(ak)Y_n(rk) \quad (3)$$

$$\sigma_n = \begin{cases}1, & n=0\\ 2, & n\neq 0\end{cases}$$

$$k^2 = \omega^2\mu\epsilon$$

As a proof of (2), consider

$$F_n(k) = \frac{ka\pi J_n(rk)f(kr_0,ka)}{4J_n'(ka)} + \frac{\delta_n}{ka} \quad (4)$$

$$0 \leqslant r \leqslant r_0 \leqslant a \qquad r_0 \neq 0$$

where

$$\delta_n = \begin{cases}1, & n=0\\ 0, & n\neq 0\end{cases}.$$

Note that no restriction is placed on n; it can be irrational, complex, etc. Now using Mittag–Leffler's expansion theorem:

$$F(z) = F(0) + \sum_{m=1}^{\infty} R_m\left\{\frac{1}{z-p_m} + \frac{1}{p_m}\right\} \quad (5)$$

where p_m is the mth pole of $F(z)$, and R_m is the residue of $F(z)$ at the mth pole. Now $F_n(k)$ has poles at $k=\pm k_{nm}$ and $F_n(0)=0$. Therefore

$$F_n(k) = F_n(0) + \sum_{m=1}^{\infty} R_m\left\{\frac{2k}{k^2-k_{nm}^2}\right\} \quad (6)$$

$$R_m = \lim_{k\to k_{nm}} \frac{ka\pi J_n(rk)f(r_0k,ak)(k-k_{nm})}{4J_n'(ka)} + \frac{(k-k_{nm})\delta_n}{ka}. \quad (7)$$

Hence

$$R_m = \frac{k_{nm}\pi J_n(rk_{nm})f(r_0k_{nm},ak_{nm})}{4J_n''(ak_{nm})}. \quad (8)$$

But

$$J_n''(ak_{nm}) = \frac{J_n(ak_{nm})}{a^2}\left(\frac{n^2}{k_{nm}^2}-a^2\right)$$

and

$$f(r_0k_{nm},ak_{nm}) = \frac{2J_n(r_0k_{nm})}{\pi ak_{nm}J_n(ak_{nm})}$$

giving

$$R_m = \frac{-aJ_n(rk_{nm})J_n(r_0k_{nm})}{2\left(a^2-\frac{n^2}{k_{nm}^2}\right)J_n^2(ak_{nm})}.$$

Therefore

$$F_n(k) = \sum_{m=1}^{\infty}\frac{kaJ_n(r_0k_{nm})J_n(rk_{nm})}{\left(a^2-\frac{n^2}{k_{nm}^2}\right)(k_{nm}^2-k^2)J_n^2(ak_{nm})}. \quad (9)$$

Hence

$$\frac{J_n(rk)f(kr_0,ka)}{4J_n'(ka)} = -\frac{\delta_n}{\pi k^2a^2} + \sum_{m=1}^{\infty}\frac{J_n(r_0k_{nm})J_n(rk_{nm})}{\left(a^2-\frac{n^2}{k_{nm}^2}\right)(k_{nm}^2-k^2)J_n^2(ak_{nm})}.$$

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Multiplying through by $j\omega\mu d\sigma_n \cos n(\phi-\phi_0)$ and then summing over n gives (2). For the special case $r=r_0=0$, the Green's function is given by

$$G(0,\phi/0,\phi_0)=\frac{-j\omega\mu d}{\pi k^2a^2}+\frac{j\omega\mu d}{\pi a^2}\sum_{m=1}^{\infty}\frac{1}{(k_{0m}^2-k^2)J_0^2(ak_{0m})}$$

which simplifies to

$$G(0,\phi/0,\phi_0)=\frac{j\omega\mu d}{4}\frac{Y_1(ak)}{J_1(ak)}. \tag{10}$$

As a proof of (10), consider

$$F(k)=\frac{akY_0'(ak)}{4J_0'(ak)}+\frac{1}{\pi ak}. \tag{11}$$

Now $F(k)$ has poles at $k=\pm k_{0m}$ and $F(0)=0$. Therefore using Mittag-Leffler's expansion gives

$$F(k)=\sum_{m=1}^{\infty}R_m\left\{\frac{2k}{k^2-k_{0m}^2}\right\} \tag{12}$$

where

$$R_m=\frac{ak_{0m}Y_0'(ak_{0m})}{4aJ_0''(ak_{0m})}=\frac{k_{0m}Y_0'(ak_{0m})}{-4J_0(ak_{0m})}=\frac{-1}{2\pi aJ_0^2(ak_{0m})}.$$

Hence

$$F(k)=\sum_{m=1}^{\infty}\frac{-2k}{2\pi a(k^2-k_{0m}^2)J_0^2(ak_{0m})}. \tag{13}$$

Subtracting from both sides $1/\pi ka$ and then multiplying by $j\omega\mu d/ak$ gives (10).

III. Green's Function for an Annular Ring

The Green's function of an annular ring microstrip antenna of inner and outer radii a and b respectively is given by [1]

$$G(r,\phi/r_0,\phi_0)=\frac{-j\omega\mu d}{k^2\pi(b^2-a^2)}+\frac{j\omega\mu d}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\sigma_n f(rk_{nm},ak_{nm})}{(k_{nm}^2-k^2)}\cdot\frac{f(r_0k_{nm},ak_{nm})\cos[n(\phi-\phi_0)]}{\left[\left(b^2-\frac{n^2}{k_{nm}^2}\right)f^2(bk_{nm},ak_{nm})-\left(a^2-\frac{n^2}{k_{nm}^2}\right)\left(\frac{\pi}{2ak_{nm}}\right)^2\right]}. \tag{14}$$

This simplifies to

$$G(r,\phi/r_0,\phi_0)=\frac{j\omega\mu d}{4}\sum_{n=0}^{\infty}\frac{\sigma_n f(rk,ak)f(r_0k,bk)\cos[n(\phi-\phi_0)]}{f'(bk,ak)} \tag{15a}$$

$$a\leqslant r\leqslant r_0\leqslant b. \tag{15b}$$

As a proof of (15a), consider

$$F_n(k)=\frac{\pi kf(rk,ak)f(r_0k,bk)}{4f'(bk,ak)}+\frac{\delta_n}{k(b^2-a^2)} \tag{16}$$
$$a\leqslant r\leqslant r_0\leqslant b$$

where

$$f'(bk,ak)=Y_n'(ak)J_n'(bk)-J_n'(ak)Y_n'(bk). \tag{17}$$

In the same manner as above, $F_n(k)$ can be expanded using Mittag-Leffler's expansion theorem as follows. $F_n(k)$ has poles at $k=\pm k_{nm}$ and $F_n(0)=0$. Hence

$$F_n(k)=\sum_{m=1}^{\infty}R_m\left\{\frac{2k}{k^2-k_{nm}^2}\right\}. \tag{18}$$

And R_m is given by

$$R_m=\lim_{k\to k_{nm}}\frac{k\pi f(rk,ak)f(r_0k,bk)(k-k_{nm})}{4f'(kb,ka)}+\frac{(k-k_{nm})\delta_n}{k(b^2-a^2)}. \tag{19}$$

Hence

$$R_m=\frac{k_{nm}\pi f(rk_{nm},ak_{nm})f(r_0k_{nm},ak_{nm})}{4f''(bk_{nm},ak_{nm})\frac{J_n'(ak_{nm})}{J_n'(bk_{nm})}} \tag{20}$$

where

$$f''(bk,ak)=\frac{\partial f'(bk,ak)}{\partial k}. \tag{21}$$

With some manipulation and the fact that

$$\mathscr{F}_n''(x)=-\frac{1}{x}\mathscr{F}_n'(x)-\frac{(x^2-n^2)}{x^2}\mathscr{F}_n(x) \tag{22}$$

we have

$$R_m=\frac{-\frac{1}{2}f(rk_{nm},ak_{nm})f(r_0k_{nm},ak_{nm})}{\left(b^2-\frac{n^2}{k_{nm}^2}\right)f^2(bk_{nm},ak_{nm})-\left(a^2-\frac{n^2}{k_{nm}^2}\right)\left(\frac{\pi}{2ak_{nm}}\right)^2}. \tag{23}$$

Hence

$$F_n(k)=\sum_{m=1}^{\infty}\frac{kf(rk_{nm},ak_{nm})f(r_0k_{nm},ak_{nm})}{(k_{nm}^2-k^2)\left[\left(b^2-\frac{n^2}{k_{nm}^2}\right)f^2(bk_{nm},ak_{nm})-\left(a^2-\frac{n^2}{k_{nm}^2}\right)\left(\frac{\pi}{2ak_{nm}}\right)^2\right]}. \tag{24}$$

Multiplying through by $j\omega\mu d\cos n(\phi-\phi_0)/\pi k$ and then summing over n gives (15a).

IV. Impedance Formulation

The elements of the impedance matrix for a multiport circular disk are given by

$$Z_{ij}=\frac{1}{W_iW_j}\int_{W_i}\int_{W_j}G(s/s_0)\,ds_0\,ds. \tag{25}$$

Hence using the double series form, the impedance matrix elements are given as

$$Z_{ij} = -\frac{j\omega\mu d\Delta_i\Delta_j}{4\pi k^2 a^2 \frac{W_i W_j}{2d_i 2d_j}} + \frac{j\omega\mu d}{4\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \cdot \frac{\sigma_n J_n(d_i k_{nm}) J_n(d_j k_{nm}) \cos n\phi_{ij} \sin n\Delta_i \sin n\Delta_j}{\left(a^2 - \frac{n^2}{k_{nm}^2}\right)(k_{nm}^2 - k^2) J_n^2(ak_{nm}) \frac{nW_i nW_j}{2d_i 2d_j}} \tag{26}$$

whereas the reduced form gives

$$Z_{ij} = \frac{j\omega\mu d}{16}\sum_{n=0}^{\infty} \frac{\sigma_n J_n(kd_i) f(kd_j, ka) \cos n(\phi_{ij}) \sin n\Delta_i \sin n\Delta_j}{J_n'(ka)\frac{nW_i nW_j}{2d_i 2d_j}} \tag{27}$$

$$0 \leqslant d_i \leqslant d_j \leqslant a, \qquad d_j \neq 0$$

where

$$\Delta_i = \sin^{-1}\left(\frac{W_i}{2d_i}\right) \qquad \phi_{ij} = \phi_i - \phi_j.$$

In the above, it is assumed that the port connections are circular.

The elements of the impedance matrix for a multiport annular ring are obtained in a similar manner, giving, for the double series form,

$$Z_{ij} = \frac{-j\omega\mu d\Delta_i\Delta_j}{4k^2\pi(b^2 - a^2)\frac{W_i W_j}{2d_i 2d_j}} + \frac{j\omega\mu d}{4\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\sigma_n f(d_i k_{nm}, ak_{nm})}{(k_{nm}^2 - k^2)} \cdot \frac{f(d_j k_{nm}, ak_{nm}) \cos n\phi_{ij} \sin n\Delta_i \sin n\Delta_j}{\left[\left(b^2 - \frac{n^2}{k_{nm}^2}\right) f^2(bk_{nm}, ak_{nm}) - \left(a^2 - \frac{n^2}{k_{nm}^2}\right)\left(\frac{\pi}{2ak_{nm}}\right)^2\right]\frac{nW_i nW_j}{2d_i 2d_j}} \tag{28}$$

and, for the single series form,

$$Z_{ij} = \frac{j\omega\mu d}{16} \cdot \sum_{n=0}^{\infty}\frac{\sigma_n f(d_i k, ak) f(d_j k, bk) \cos n\phi_{ij} \sin n\Delta_i \sin n\Delta_j}{f'(bk, ak)\frac{nW_i nW_j}{2d_i 2d_j}} \tag{29}$$

$$a \leqslant d_i \leqslant d_j \leqslant b.$$

For the case where the ports are around the periphery the equations can be simplified further as follows. The double-series form for the disk gives

$$Z_{ij} = \frac{-j\omega\mu d}{4\pi k^2 a^2} + \frac{j\omega\mu d}{4\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\sigma_n \cos n\phi_{ij} \operatorname{sinc}^2\left(\frac{nW}{2a}\right)}{\left(a^2 - \frac{n^2}{k_{nm}^2}\right)(k_{nm}^2 - k^2)} \tag{30}$$

and the reduced form gives

$$Z_{ij} = \frac{j\omega\mu d}{8\pi ka}\sum_{n=0}^{\infty}\frac{\sigma_n J_n(ka) \cos n\phi_{ij} \operatorname{sinc}^2\left(\frac{nW}{2a}\right)}{J_n'(ka)}. \tag{31}$$

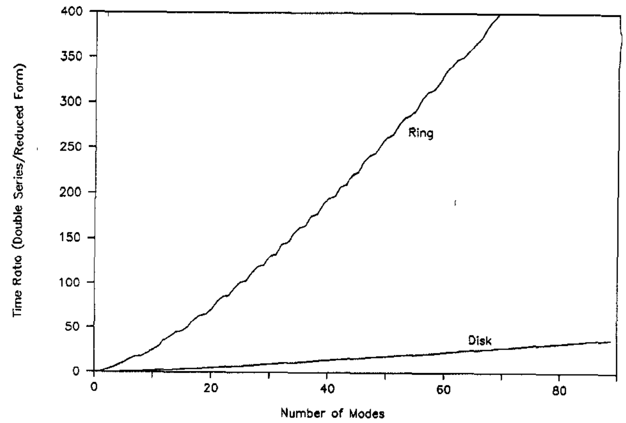

Fig. 1. Timing for disk and ring.

The double-series form for the ring gives

$$Z_{ij} = \frac{-j\omega\mu d}{4k^2\pi(b^2 - a^2)} + \frac{j\omega\mu d}{4\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\sigma_n f^2(bk_{nm}, ak_{nm})}{(k_{nm}^2 - k^2)} \cdot \frac{\cos n\phi_{ij} \operatorname{sinc}^2\left(\frac{nW}{2b}\right)}{\left[\left(b^2 - \frac{n^2}{k_{nm}^2}\right) f^2(bk_{nm}, ak_{nm}) - \left(a^2 - \frac{n^2}{k_{nm}^2}\right)\left(\frac{\pi}{2ak_{nm}}\right)^2\right]} \tag{32}$$

and the reduced form gives

$$Z_{ij} = \frac{j\omega\mu d}{8\pi kb}\sum_{n=0}^{\infty}\frac{\sigma_n f(bk, ak) \cos n\phi_{ij} \operatorname{sinc}^2\left(\frac{nW}{2b}\right)}{f'(bk, ak)} \tag{33}$$

where

$$\Delta_i \approx \frac{W}{2b}.$$

V. Comparison of the Two Forms

Comparison of the equivalence and time taken for the computation of the two forms was carried out by simulating two junctions: (i) A 4-port rat race circular disk of radius 15 mm and (ii) a 4-port annular ring of outer and inner radii 15 mm and 5 mm respectively. In both cases the ports widths were 4.4 mm and were positioned at 0°, 90°, 135°, and 225°. The substrate was assumed to have a relative permittivity of $\epsilon_r = 2.5$ and a height of 1.524 mm.

The results are given for Z_{11} at a frequency of 4.5 (GHz). Fig. 1 shows the time ratios (time taken by double series/time taken by reduced form) against the number of modes used for the annular ring and the circular disk. Fig. 2 shows the values of Z_{11}

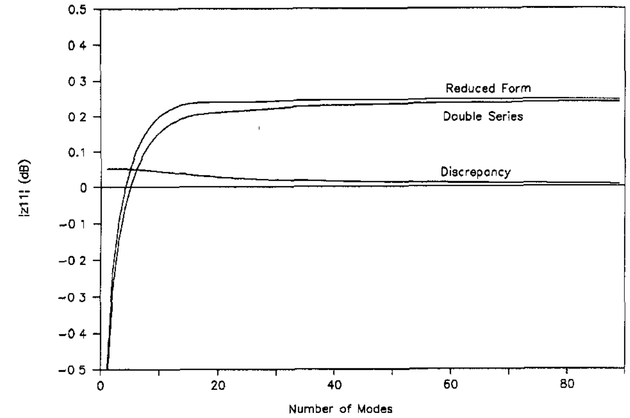

Fig. 2. Z_{11} for disk.

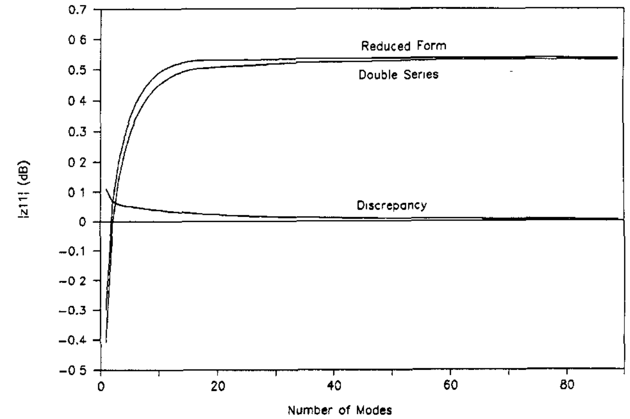

Fig. 3. Z_{11} for ring.

for the circular disk for both the double series and the reduced form and the discrepancy between them. Fig. 3 shows the values of Z_{11} for the annular ring for both the double series and the reduced form as well as the discrepancy between them.

VI. Conclusion

The Green's functions for the circular disk and the annular ring have been reduced to single-series forms in a mathematically direct manner, eliminating the need for the eigenvalues and, as a consequence, improving the speed and accuracy of the computation.

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A Surface Integral Equation Method for the Finite Element Solution of Waveguide Discontinuity Problems

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***Abstract*—The surface integral equation method, which is typically employed in the finite element solution of open-region scattering problems, has been applied in this paper to the solution of waveguide discontinuity problems. The major advantage offered by the surface integral equation approach over other available methods is that it allows the mesh-truncating boundaries to be brought as close to the discontinuity as possible, thus helping to reduce the size of the system matrix. In addition, unlike the mode matching technique, the surface integral equation formulation does not require the solution of any auxiliary matrix system. Numerical results are presented to illustrate the validity of the formulation.**

I. Introduction

When designing waveguide devices, it is often necessary to introduce discontinuities or loads that are used for different purposes such as phase shifting or power matching to a specific load or termination. The analysis of such waveguide junctions or discontinuities has traditionally been carried out using the mode-matching techniques and the integral equation method [1], [2]. However, when the discontinuities are irregularly shaped or involve inhomogeneous or anisotropic objects, the integral equation methods become quite laborious and difficult to apply. For such complex and irregularly shaped geometries, either the finite element or the finite difference method becomes the preferred choice. Additionally, the finite methods generate highly sparse and banded matrices which can be efficiently handled using special algorithms.

When using the finite element (or the finite difference) method to solve boundary values problems such as waveguide discontinuities, two major consideration arise. First, it is always desirable to bring the mesh-truncating boundary as close as possible to the discontinuity junction in order to reduce the number of mesh points and, hence, the size of the associated matrix. Second, a boundary condition must be imposed on the terminal boundaries such that it accurately reflects the proper field behavior there. The task of devising an efficient solution procedure that accommodates the above two considerations is the principal subject of discussion in this paper.

Conventionally, finite element formulations of the waveguide discontinuity problem are based upon the truncation of the finite element mesh region at a distance where the amplitudes of the evanescent modes become negligible, and then the imposition of a Dirichlet or a Neumann boundary condition [3], [4]. Such construction offers the advantage of generating a sparse matrix system. In certain applications, such as the modeling of electromagnetic pulse simulators, the width of the simulator/waveguide may range from a fraction of a wavelength to tens of

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